

# Locally maximal product-free sets of size 3

By

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## Abstract

Let  $G$  be a group, and  $S$  a non-empty subset of  $G$ . Then  $S$  is *product-free* if  $ab \notin S$  for all  $a, b \in S$ . We say  $S$  is *locally maximal product-free* if  $S$  is product-free and not properly contained in any other product-free set. A natural question is what is the smallest possible size of a locally maximal product-free set in  $G$ . The groups containing locally maximal product-free sets of sizes 1 and 2 were classified in [3]. In this paper, we prove a conjecture of Giudici and Hart in [3] by showing that if  $S$  is a locally maximal product-free set of size 3 in a group  $G$ , then  $|G| \leq 24$ . This shows that the list of known locally maximal product-free sets given in [3] is complete.

## 1 Introduction

Let  $G$  be a group, and  $S$  a non-empty subset of  $G$ . Then  $S$  is *product-free* if  $ab \notin S$  for all  $a, b \in S$ . For example, if  $H$  is a subgroup of  $G$  then  $Hg$  is a product-free set for any  $g \notin H$ . Traditionally these sets have been studied in abelian groups, and have therefore been called sum-free sets. Since we are working with arbitrary groups it makes more sense to say ‘product-free’ in this context. We say  $S$  is *locally maximal product-free* if  $S$  is product-free and not properly contained in any other product-free set. We use the term *locally maximal* rather than maximal because the majority of the literature in this area uses *maximal* to mean maximal by cardinality (for example [7, 8]).

There are some obvious questions from the definition: given a group  $G$ , what is the maximum cardinality of a product-free set in  $G$ , and what are the maximal (by cardinality) product-free sets? How many product-free sets are there in  $G$ ? Given that each product-free set is contained in a locally maximal product-free set, what are the locally maximal product-free sets? What are the possible sizes of locally maximal product-free sets? The question of maximal (by cardinality) product-free sets has been fully solved for abelian groups by Green and Ruzsa [5]. For the nonabelian case Kedlaya [6] showed that there exists a constant  $c$  such

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that the largest product-free set in a group of order  $n$  has size at least  $cn^{11/14}$ . Gowers [4] proved that if the smallest nontrivial representation of  $G$  is of dimension  $k$  then the largest product-free set in  $G$  has size at most  $k^{-1/3}n$  (Theorem 3.3 and commentary at the start of Section 5). Much less is known about the minimum sizes of locally maximal product-free sets. This question was first asked in [1] where the authors ask what is the minimum size of a locally maximal product-free set in a group of order  $n$ ? A good bound is still not known. Small locally-maximal product-free sets when  $G$  is an elementary abelian 2-group are of interest in finite geometry, because they correspond to complete caps in  $\text{PG}(n-1, 2)$ . In [3], the groups containing locally maximal product-free sets of sizes 1 and 2 were classified. Some general results were also obtained. Furthermore, there was a classification (Theorem 5.6) of groups containing locally maximal product-free sets  $S$  of size 3 for which not every subset of size 2 in  $S$  generates  $\langle S \rangle$ . Each of these groups has order at most 24. Conjecture 5.7 of [3] was that if  $G$  is a group of order greater than 24, then  $G$  does not contain a locally maximal product-free set of size 3. Table 5 listed all the locally maximal product-free sets in groups of orders up to 24. So the conjecture asserts that this list is the complete list of all such sets. We have reproduced Table 5 as Table 1 in this paper because we need to use it in some of the arguments here. The main result of this paper is the following and its immediate corollary.

**Theorem 1.1.** *Suppose  $S$  is a locally maximal product-free set of size 3 in a group  $G$ , such that every two element subset of  $S$  generates  $\langle S \rangle$ . Then  $|G| \leq 24$ .*

**Corollary 1.2.** *If a group  $G$  contains a locally maximal product-free set  $S$  of size 3, then  $|G| \leq 24$  and the only possibilities for  $G$  and  $S$  are listed in Table 1.*

*Proof.* If not every two-element subset of  $S$  generates  $\langle S \rangle$ , then by Theorem 5.6 of [3],  $|G| \leq 24$ . We may therefore assume that every two-element subset of  $S$  generates  $\langle S \rangle$ . Then  $|G| \leq 24$  by Theorem 1.1. Now Table 1 is just Table 5 of [3]; it is a list of all locally maximal product-free sets of size 3 occurring in groups of order up to 24 (in fact, up to 37 in the original paper). Since we have shown that all locally maximal product-free sets of size 3 occur in groups of order up to 24, this table now constitutes a complete list of possibilities.  $\square$

We finish this section by establishing the notation to be used in the rest of the paper, and giving some basic results from [3]. For subsets  $A, B$  of a group  $G$ , we use the standard notation  $AB$  for the product of  $A$  and  $B$ . That is,

$$AB = \{ab : a \in A, b \in B\}.$$

By definition, a nonempty set  $S \subseteq G$  is product-free if and only if  $S \cap SS = \emptyset$ . In order to investigate locally maximal product-free sets, we introduce some further notations. For a

set  $S \subseteq G$ , we define the following sets:

$$\begin{aligned} S^2 &= \{a^2 : a \in S\}; \\ S^{-1} &= \{a^{-1} : a \in S\}; \\ \sqrt{S} &= \{x \in G : x^2 \in S\}; \\ T(S) &= S \cup SS \cup SS^{-1} \cup S^{-1}S; \\ \hat{S} &= \{s \in S : \sqrt{\{s\}} \not\subset \langle S \rangle\}. \end{aligned}$$

For a singleton set  $\{a\}$ , we usually write  $\sqrt{a}$  instead of  $\sqrt{\{a\}}$ .

For a positive integer  $n$ , we will denote by  $\text{Alt}(n)$  the alternating group of degree  $n$ , by  $C_n$  the cyclic group of order  $n$ , by  $D_{2n}$  the dihedral group of order  $2n$ , and by  $Q_{4n}$  the dicyclic group of order  $4n$  given by  $Q_{4n} := \langle x, y : x^{2n} = 1, x^n = y^2, yx = x^{-1}y \rangle$ .

We finish this section with a few results from [3].

**Lemma 1.3.** [3, Lemma 3.1] *Suppose  $S$  is a product-free set in the group  $G$ . Then  $S$  is locally maximal product-free if and only if  $G = T(S) \cup \sqrt{S}$ .*

The next result lists, in order, Proposition 3.2, Theorem 3.4, Propositions 3.6, 3.7, 3.8 and Corollary 3.10 of [3].

**Theorem 1.4.** *Let  $S$  be a locally maximal product-free set in a group  $G$ . Then*

- (i)  $\langle S \rangle$  is a normal subgroup of  $G$  and  $G/\langle S \rangle$  is either trivial or an elementary abelian 2-group;
- (ii)  $|G| \leq 2|T(S)| \cdot |\langle S \rangle|$ ;
- (iii) if  $\langle S \rangle$  is not an elementary abelian 2-group and  $|\hat{S}| = 1$ , then  $|G| = 2|\langle S \rangle|$ ;
- (iv) every element  $s$  of  $\hat{S}$  has even order, and all odd powers of  $s$  lie in  $S$ ;
- (v) if there exists  $s \in S$  and integers  $m_1, \dots, m_t$  such that  $\hat{S} = \{s, s^{m_1}, \dots, s^{m_t}\}$ , then  $|G|$  divides  $4|\langle S \rangle|$ ;
- (vi) if  $S \cap S^{-1} = \emptyset$ , then  $|G| \leq 4|S|^2 + 1$ .

We require one final result.

**Theorem 1.5.** [3, Theorem 5.1] *Up to isomorphism, the only instances of locally maximal product-free sets  $S$  of size 3 of a group  $G$  where  $|G| \leq 37$  are given in Table 1.*

## 2 Proof of Theorem 1.1

**Proposition 2.1.** *Suppose  $S$  is locally maximal product-free of size 3 in  $G$ . If  $\langle S \rangle$  is cyclic, then  $|G| \leq 24$ .*

*Proof.* Write  $S = \{a, b, c\}$ . First note that since  $\langle S \rangle$  is abelian,  $SS^{-1} = S^{-1}S$ ; moreover  $aa^{-1} = bb^{-1} = cc^{-1} = 1$ ; so  $|SS^{-1}| \leq 7$ . Also  $SS \subseteq \{a^2, b^2, c^2, ab, ac, bc\}$ . Thus

$$|T(S)| = |S \cup SS \cup SS^{-1}| \leq 3 + 6 + 7 = 16.$$

By Lemma 1.3,  $G = T(S) \cup \sqrt{S}$ ; so  $\langle S \rangle = T(S) \cup (\langle S \rangle \cap \sqrt{S})$ . Elements of cyclic groups have at most two square roots. Therefore  $|\langle S \rangle| \leq 16 + 6 = 22$ . By Table 1,  $\langle S \rangle$  must now be one of  $C_6, C_8, C_9, C_{10}, C_{11}, C_{12}, C_{13}$  or  $C_{15}$ . Theorem 1.4(iv) tells us that every element  $s$  of  $\hat{S}$  has even order and all odd powers of  $s$  lie in  $S$ . This means that for  $C_9, C_{11}, C_{13}$  or  $C_{15}$ , we have  $\hat{S} = \emptyset$  and so  $G = \langle S \rangle$ . In particular,  $|G| \leq 24$ .

It remains to consider  $C_6, C_8, C_{10}$  and  $C_{12}$ . For  $C_6 = \langle g : g^6 = 1 \rangle$ , the unique locally maximal product-free set of size 3 is  $S = \{g, g^3, g^5\}$ . Now if  $g$  or  $g^5$  is contained in  $\hat{S}$ , then  $\hat{S}$  consists of powers of a single element; so by Theorem 1.4(v),  $|G|$  divides 24. If neither  $g$  nor  $g^5$  is in  $\hat{S}$ , then  $|\hat{S}| \leq 1$ , and so by Theorem 1.4(iii) therefore,  $|G|$  divides 12. In  $C_8$  there is a unique (up to group automorphisms) locally maximal product-free set of size 3, and it is  $\{g, g^{-1}, g^4\}$ , where  $g$  is any element of order 8. If  $\hat{S}$  contains  $g$  or  $g^{-1}$ , then  $S$  contains all odd powers of that element by Theorem 1.4(iv), and hence  $S$  contains  $\{g, g^3, g^5, g^7\}$ , a contradiction. Therefore  $|\hat{S}| \leq 1$  and so  $|G|$  divides 16. Next, we consider  $\langle S \rangle = C_{10}$ . Recall that elements of  $\hat{S}$  must have even order. If  $\hat{S}$  contains any element of order 10, then  $S$  contains all five odd powers of this element, which is impossible by Theorem 1.4(iv). This leaves only the involution of  $C_{10}$  as a possible element of  $\hat{S}$ . Hence again  $|\hat{S}| \leq 1$  and  $|G|$  divides 20. Finally we look at  $C_{12}$ . If  $\hat{S}$  contains any element of order 12, then  $|S| \geq 6$ , a contradiction. If  $\hat{S}$  contains an element  $x$  of order 6 then  $S$  contains all three of its odd powers, so  $S = \{x, x^3, x^5\}$ . But then  $\langle S \rangle \cong C_6$ , contradicting the assumption that  $\langle S \rangle = C_{12}$ . Therefore,  $\hat{S}$  can only contain elements of order 2 or 4. Up to group automorphism, we see from Table 1 that every locally maximal product-free set  $S$  of size 3 in  $C_{12}$  with  $\langle S \rangle = C_{12}$  is one of  $\{g, g^6, g^{10}\}$  or  $\{g, g^3, g^8\}$  for some generator  $g$  of  $C_{12}$ . Each of these sets contains exactly one element of order 2 or 4. Therefore in every case,  $|\hat{S}| \leq 1$  and so  $|G|$  divides 24. This completes the proof.  $\square$

Note that the bound on  $|G|$  in Proposition 2.1 is attainable. For example in  $Q_{24}$  there is a locally maximal product-free set  $S$  of size 3, with  $\langle S \rangle \cong C_{12}$ .

**Proposition 2.2.** *Suppose  $S$  is locally maximal product-free of size 3 in  $G$  such that every 2-element subset of  $S$  generates  $\langle S \rangle$ . Then either  $|G| \leq 24$  or  $S$  contains exactly one involution.*

*Proof.* First suppose  $S$  contains no involutions. If  $S \cap S^{-1} = \emptyset$ , then Theorem 1.4(vi) tells us that  $G$  has order at most 37, and then by Theorem 1.5,  $(G, S)$  is one of the possibilities listed in Table 1. In particular  $|G| \leq 24$ . If  $S \cap S^{-1} \neq \emptyset$ , then  $S = \{a, a^{-1}, b\}$  for some  $a, b$ .

But then  $\langle S \rangle = \langle a, a^{-1} \rangle = \langle a \rangle$ , so  $\langle S \rangle$  is cyclic. Now by Proposition 2.1 we get  $|G| \leq 24$ . Next, suppose that  $S$  contains at least two involutions,  $a$  and  $b$ , with the third element being  $c$ . Then, since every 2-element subset of  $S$  generates  $\langle S \rangle$ , we have that  $H = \langle S \rangle = \langle a, b \rangle$  is dihedral and  $S$  is locally maximal product-free in  $H$ . Let  $o(ab) = m$ , so  $H \cong D_{2m}$ . The non-trivial coset of the subgroup  $\langle ab \rangle$  is product-free of size  $m$ . So if  $c$  lies in this coset, then we have  $m = 3$  and  $H \cong D_6$ . If  $c$  does not lie in this coset then  $c = (ab)^i$  for some  $i$ , and from the relations in a dihedral group  $ac^{-1} = ca$ ,  $c^{-1}a = ac$ ,  $bc^{-1} = cb$  and  $c^{-1}b = bc$ . The coset  $\langle ab \rangle a$  consists of  $m$  involutions, which cannot lie in  $\sqrt{S}$ . Thus  $\langle ab \rangle a \subseteq T(S)$  by Lemma 1.3. A straightforward calculation shows that

$$\begin{aligned} \langle ab \rangle a = T(S) \cap \langle ab \rangle a &= \{a, b, ac, ca, bc, cb, ac^{-1}, c^{-1}a, bc^{-1}, c^{-1}b\} \\ &= \{a, b, ac, ca, bc, cb\} \end{aligned}$$

This means  $m \leq 6$ , and  $S$  consists of two generating involutions  $a, b$  plus a power of their product  $ab$ , with the property that any two-element subset of  $S$  generates  $\langle a, b \rangle$ . A glance at Table 1 shows there are no locally maximal product-free sets of this form in  $D_{2m}$  for  $m \leq 6$ . Therefore the only possibility is that  $\langle S \rangle \cong D_6$ , with  $S$  consisting of the three reflections in  $\langle S \rangle$ . By Theorem 1.4(i), the index of  $\langle S \rangle$  in  $G$  is a power of 2. By Theorem 1.4(ii),  $|G| \leq 2|T(S)| \cdot |\langle S \rangle|$ . Thus  $|G| \in \{6, 12, 24, 48\}$ . Suppose for contradiction that  $|G| = 48$ . Now  $G = T(S) \cup \sqrt{S}$ , and since  $S$  consists of involutions, the elements of  $\sqrt{S}$  have order 4. So  $G$  contains two elements of order 3, three elements of order 2 and the remaining non-identity elements have order 4. Then the 46 elements of  $G$  whose order is a power of 2 must lie in three Sylow 2-subgroups of order 16, with trivial pairwise intersection. Each of these groups therefore has a unique involution and 14 elements of order 4, all of which square to the given involution. But no group of order 16 has fourteen elements of order 4. Hence  $|G| \neq 48$ , and so  $|G| \leq 24$ . Therefore either  $|G| \leq 24$  or  $G$  contains exactly one involution.  $\square$

Before we establish the next result, we first make a useful observation. Suppose  $S = \{a, b, c\}$  where  $a, b, c \in G$  and  $c$  is an involution. Then a straightforward calculation shows that

$$T(S) \subseteq \left\{ \begin{array}{l} 1, a, b, c, a^2, b^2, ab, ba, ac, ca, bc, cb, \\ ab^{-1}, ba^{-1}, ca^{-1}, cb^{-1}, a^{-1}b, a^{-1}c, b^{-1}a, b^{-1}c \end{array} \right\}. \quad (1)$$

**Lemma 2.3.** *Suppose  $S$  is a locally maximal product-free set of size 3 in  $G$ , every 2-element subset of  $S$  generates  $\langle S \rangle$ , and  $S$  contains exactly one involution. Then either  $|G| \leq 24$  or  $S = \{a, b, c\}$ , where  $a$  and  $b$  have order 3 and  $c$  is an involution.*

*Proof.* Suppose  $S = \{a, b, c\}$  where  $c$  is an involution and  $a, b$  are not. Consider  $a^{-1}$ . Recall that  $G = T(S) \cup \sqrt{S}$ . If  $a^{-1} \in \sqrt{S}$  then  $a^{-2} \in \{a, b, c\}$  which implies that either  $a$  has order 3 or  $\langle S \rangle$  is cyclic (because for example if  $a^{-2} = b$  then  $\langle S \rangle = \langle a, b \rangle = \langle a \rangle$ ). Thus if  $a^{-1} \in \sqrt{S}$  implies that either  $a$  has order 3 or (by Lemma 2.1)  $|G| \leq 24$ . Suppose then that  $a^{-1} \in T(S)$ . The elements of  $T(S)$  are given in Equation 1. If  $a^{-1} \in \{b, b^2, ab, ba, ab^{-1}, ba^{-1}, a^{-1}b, b^{-1}a\}$  then by remembering that  $\langle S \rangle = \langle a, b \rangle$ , we deduce that  $\langle S \rangle$  is cyclic, generated by either  $a$  or  $b$ . For example,  $a^{-1} = ba$  implies  $b \in \langle a \rangle$ . Similarly, if  $a^{-1} \in \{c, ac, ca, a^{-1}c, c^{-1}a\}$ , then  $\langle S \rangle$  is cyclic. Since  $a$  has order at least 3, we cannot have  $a^{-1} \in \{1, a\}$ . If  $a^{-1} \in \{bc, cb, b^{-1}c, c^{-1}b\}$ ,

then  $S$  would not be product-free. For instance  $a^{-1} = b^{-1}c$  implies that  $b^{-1}ca = 1$ , and hence  $ac = b$ . The only remaining possibility is  $a^{-1} = a^2$ , meaning that  $a$  has order 3. The same argument with  $b^{-1}$  shows that  $b$  also has order 3.  $\square$

We can now prove Theorem 1.1, which states that if  $S$  is a locally maximal product-free set of size 3 in a group  $G$ , such that every two element subset of  $S$  generates  $\langle S \rangle$ , then  $|G| \leq 24$ .

**Proof of Theorem 1.1** Suppose  $S$  is a locally maximal product-free set of size 3 in  $G$  such that every two element subset of  $S$  generates  $\langle S \rangle$ . Then by Lemma 2.3, either  $|G| \leq 24$  or  $S = \{a, b, c\}$  where  $a$  and  $b$  have order 3 and  $c$  is an involution. In the latter case, we observe that  $aca^{-1}$  is an involution, so must be contained in  $T(S)$ . Using Equation 1 we work through the possibilities. Obviously it is impossible for  $aca^{-1}$  to be equal to any of  $1, a, b, a^2$  or  $b^2$  because these elements are not of order 2. If any of  $ac, ca, a^{-1}c, c^{-1}a, bc, cb, b^{-1}c$  or  $cb^{-1}$  were involutions, then it would imply that  $\langle S \rangle$  was generated by two involutions whose product has order 3. For example if  $ac$  were an involution then  $\langle c, ac \rangle = \langle a, c \rangle = \langle S \rangle$ . That is,  $\langle S \rangle$  would be dihedral of order 6. But there is no product-free set in  $D_6$  containing two elements of order 3, because if  $x, y$  are the elements of order 3 in  $D_6$  then  $x^2 = y$  and  $y^2 = x$ . So the remaining possibilities for  $aca^{-1}$  are  $c, ab, ba, ab^{-1}, ba^{-1}, a^{-1}b$  and  $b^{-1}a$ . Now  $aca^{-1} = ab$  implies  $c = ba$ , whereas  $aca^{-1} = ab^{-1}$  implies  $bc = a$  and  $aca^{-1} = ba^{-1}$  implies  $b = ac$ , each of which contradicts the fact that  $S$  is product-free. We are now left with the cases  $aca^{-1} = c, aca^{-1} = ba$  and  $aca^{-1} = a^{-1}b$  (which, if it is an involution, equals  $b^{-1}a$ ). If  $aca^{-1} = c$ , then  $\langle S \rangle = \langle a, c \rangle = C_6$ , but the only product-free set of size 3 in  $C_6$  contains no elements of order 3, so this is impossible. Therefore  $aca^{-1} \in \{ba, a^{-1}b\}$ . If  $aca^{-1} = ba$ , then  $a^{-1}ba = ca^{-1}$ , so  $ac = a^{-1}b^{-1}a$ , which has order 3. If  $aca^{-1} = a^{-1}b$ , then  $ac = a^{-1}ba$ , again of order 3. So we see that

$$\langle S \rangle = \langle a, c : a^3 = 1, c^2 = 1, (ac)^3 = 1 \rangle.$$

This is a well known presentation of the alternating group  $\text{Alt}(4)$ . As  $c$  is the only element of  $S$  whose order is even, we see that  $|\hat{S}| \leq 1$ , and hence  $|G| \leq 2|\text{Alt}(4)| = 24$ . Therefore in all cases  $|G| \leq 24$ .  $\square$

### 3 Data and Programs

Though Table 1 is essentially just Table 5 from [3], we have taken the opportunity here to correct a typographical error in the entry for the (un-named) group of order 16. We provide below the GAP programs used to obtain the table.

**Program 3.1.** *A program that tests if a set  $T$  is product-free.*

```
## It returns "0" if T is product-free, and "1" if otherwise.
prodtest:= function(T)
local x, y, prod;
prod:=0;
```

```

for x in T do
  for y in T do
    if x*y in T then
      prod:=1;
    fi;
  od;
od;
return prod;
end;

```

**Program 3.2.** *A program for finding all locally maximal product-free sets of size 3 in G.*

```

##It prints the list of all locally maximal product-free sets of size 3 in G.
LMPFS3:=function(G)
local L, lmpf, combs, x, pf, H, y, z, s, i, q;
L:=AsSortedList(G); lmpf=[]; combs:=Combinations(L,3);
for i in [1..Binomial(Size(L),3)] do
  pf:=combs[i];
  if prodtest(pf)=0 then
    s:=Size(lmpf); H:=Difference(L,pf);
    for y in [1..3] do
      for z in [1..3] do
        H:=Difference(H, [pf[y]*pf[z], pf[y]*(pf[z])^-1, ((pf[y])^-1)*pf[z]]);
      od;
    od;
    for q in L do
      if q^2 in pf then
        H:=Difference(H, [q]);
      fi;
    od;
    if Size(H) = 0 then
      lmpf:=Union(lmpf, [pf]);
    fi;
  fi;
od;
if Size(lmpf) > 0 then
  Print(G,"\n",L,"\n","Structure Description of G is ",StructureDescription(G),
  "\n", "Gap Id of G is ", IdGroup(G), "\n", "\n", lmpf, "\n", "\n");
fi;
end;

```

$G$	$S$	$\langle S \rangle$	# Locally maximal product-free sets of size 3 in $G$
$\langle g : g^6 = 1 \rangle$	$\cong C_6$	$\cong C_6$	1
$\langle g, h : g^3 = h^2 = 1, hgh = g^{-1} \rangle$	$\cong D_6$	$\cong D_6$	1
$\langle g : g^8 = 1 \rangle$	$\cong C_8$	$\cong C_8$	2
$\langle g, h : g^4 = h^2 = 1, hgh^{-1} = g^{-1} \rangle$	$\cong D_8$	$\cong D_8$	4
$\langle g : g^9 = 1 \rangle$	$\cong C_9$	$\cong C_9$	8
$\langle g, h : g^3 = h^3 = 1, gh = hg \rangle$	$\cong C_3 \times C_3$	$\cong C_3 \times C_3$	8
$\langle g : g^{10} = 1 \rangle$	$\cong C_{10}$	$\cong C_{10}$	6
$\langle g : g^{11} = 1 \rangle$	$\cong C_{11}$	$\cong C_{11}$	10
$\langle g : g^{12} = 1 \rangle$	$\cong C_{12}$	$\cong C_{12}$	1
$\langle g, h : g^6 = 1, g^3 = h^2, hgh^{-1} = g^{-1} \rangle$	$\cong Q_{12}$	$\cong C_6$	8
Alternating group of degree 4	$= \text{Alt}(4)$	$\cong \text{Alt}(4)$	1
			48
$\langle g : g^{13} = 1 \rangle$	$\cong C_{13}$	$\cong C_{13}$	16
$\langle g : g^{15} = 1 \rangle$	$\cong C_{15}$	$\cong C_{15}$	4
$\langle g, h : g^4 = h^4 = 1, gh = hg \rangle$	$\cong C_4 \times C_4$	$\cong C_4 \times C_4$	16
$\langle g, h : g^8 = 1, g^4 = h^2, hgh^{-1} = g^{-1} \rangle$	$\cong Q_{16}$	$\cong C_8$	2
$\langle g, h : g^8 = h^2 = 1, hgh^{-1} = g^5 \rangle$	(order 16)	$\cong G$	8
$\langle g, h : g^{10} = 1, g^5 = h^2, hgh^{-1} = g^{-1} \rangle$	$\cong Q_{20}$	$\cong C_{10}$	6
$\langle g, h : g^3 = h^7 = 1, ghg^{-1} = h^2 \rangle$	$\cong C_7 \rtimes C_3$	$\cong C_7 \rtimes C_3$	42
$\langle x : x^3 = 1 \rangle \times \langle g, h : g^4 = 1, g^2 = h^2, hgh^{-1} = g^{-1} \rangle$	$\cong C_3 \times Q_8$	$\cong C_6$	1
$\langle g, h : g^{12} = 1, g^6 = h^2, hgh^{-1} = g^{-1} \rangle$	$\cong Q_{24}$	$\cong C_6$	1
		$\cong C_{12}$	4

Table 1: Locally maximal product-free sets of size 3 in groups of order up to 24

## References

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